Journal of Applied Mechanics and Technical Physics, Vol. 44, No. 4, pp. 584–595, 2003

DESIGN OF A BEAM WITH CONTROLLABLE FORCE ELEMENTS

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UDC 539.3

A method for solving the problem of design of an "intellectual" structure formulated for the pair {optimal position of actuators}, {optimal control of actuators} is developed. In the method proposed, physical and logical objects are treated as "equivalent."

Key words: *design, "intellectual" structure, actuators, controlling instruction.*

At present, attention is drawn to the so-called intellectual structures with force elements (actuators), which allow the structures to affect themselves and, as a result, adapt to external loads. The force elements are controlled by a local processor (microcomputer), which is a component of the structure. The main problem for these structures is the design problem.

The number and location of actuators and instruction (computer program that controls the behavior of actuators) are interrelated [1]; therefore, the problem is to determine simultaneously the number and location of actuators and develop a program for controlling the actuators. This formulation of the problem has been extensively discussed (see, e.g., books [2, 3] and references cited therein). However, to the author's knowledge, the solution of this problem (and even mathematical formalization) that relates all the components of an "intellectual" structure (passive elements, force elements, and controlling program) has not been obtained.

In the present paper, the problem is formulated for the pair {*optimal location of actuators*}, {*optimal control of actuators*}. If the location of actuators is specified, the problem of optimal control arises [4]. If the control is specified (a set of force elements of specified intensity is given, and the question is to find their best location), the problem is similar to the problem of topology design [5]. It is worth noting that passive elements (in this case, a beam), devices for measuring loads (sensors), and force elements (actuators) are physical objects, whereas programs are logical objects. One of the goals of this paper is to develop a method in which the physical and logical objects are considered as "equivalent" (within the framework of an appropriate mathematical formalization).

In this paper, we describe the method with reference to the simplest structure — an "intellectual" beam. The solution is obtained by reducing the starting problem to the problem of the relative location of convex polyhedrons, one of which corresponds to the set of possible deflections and the other to objective function. The method proposed can be applied to structures of various types.

1. Formulation of the Problem and Its Solution for Steady Loading. We consider a beam with the ends $y = 0$ and $y = 1$. The beam is loaded by an external force $F(y)$. Moreover, the beam can be subjected to forces and moments of intensity $p(y)$ ($y \in [0, 1]$) produced by the actuators.

Locations and intensities of the actuators are unknown in advance; there is only a restriction imposed on the intensities. It is required to determine the actuator locations and intensities (forces and moments) that minimize the beam deflection $u(x)$ in the interval $[0, 1]$. This problem is a model problem of maintaining the structure shape.

The locations and intensities of actuators are described by the function $p(y)$. If $p(y) = 0$, there are no actuators at the point y; if $p(y) \neq 0$, there is an actuator at the point y and the function $p(y)$ specifies its intensity.

Siberian State University of Telecommunications and Informatics, Novosibirsk 630102. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 44, No. 4, pp. 159–173, July–August, 2003. Original article submitted November 9, 2000; revision submitted January 14, 2002.

The beam deflection $u(x)$ produced by the force $F(y)$ and force actuators (force elements producing additional forces applied to the beam) of intensity $p(y)$ is determined from the equation

$$u^{\mathrm{IV}} = p + F \tag{1.1}$$

subject to the clamped boundary conditions

$$u(0) = u(1) = 0, \qquad u'(0) = u'(1) = 0 \tag{1.2}$$

(different boundary conditions can also be imposed).

For the moment actuators (force elements producing additional moments in the beam) of intensity $p(y)$ [1], we obtain the equation

$$(u'' + p)'' = F \tag{1.3}$$

subject to the same boundary conditions (1.2).

We consider the restriction imposed on the intensity of actuators

$$\int_0^1 p(y)\, dy \leqslant 1 \tag{1.4}$$

(total restriction on the intensity of actuators) and

$$p(y) \geqslant 0. \tag{1.5}$$

Expressions (1.4) and (1.5) are model restrictions simplifying subsequent calculations.

It is required to minimize the beam deflection $u(x)$, i.e., solve the problem

$$\|u\|_{C[0,1]} \equiv \max_{x\in[0,1]} |u(x)| \to \min. \tag{1.6}$$

In (1.6), minimization is performed for the functions $p(y)$ satisfying (1.4) and (1.5). In this case, $u(x)$ is determined from (1.1), (1.2) or (1.3), (1.2).

In the formulation considered, the class (denoted by A) to which the function $p(y)$ belongs is rather wide. In reality, actuators produce a force of a particular type. We consider the class of forces

$$A_\delta = \Big\{ \sum_{j=1}^{m} p_j \delta(x - y_j) \Big\}, \tag{1.7}$$

where $\delta(x)$ is the delta function. This class of forces corresponds to point actuators. Other classes of forces can be used, for example, the class of functions constant in some intervals corresponds to piezoelectric patches [1].

If there are no restrictions imposed on forces produced by actuators, the problem has the trivial solution $p(y) = -F(y)$. For restrictions of the form (1.4) and (1.5) or restrictions for the class A, the solution $p(x) = -F(x)$ can be inadmissible. A typical restriction on the class A which makes the solution $p(x) = -F(x)$ inadmissible is the use of a finite number of actuators.

The solutions of problems (1.1), (1.2) and (1.3), (1.2) have the form

$$u(x) = \int_0^1 L(x,y)F(y)\, dy + \int_0^1 L(x,y)p(y)\, dy, \tag{1.8}$$

$$u(x) = \int_0^1 L(x,y)F(y)\, dy + \int_0^1 M(x,y)p(y)\, dy, \tag{1.9}$$

respectively. Here $L(x,y)$ and $M(x,y)$ are the fundamental solutions of beam bending

$$L^{\mathrm{IV}} = \delta(x - y), \qquad M^{\mathrm{IV}} = -\delta''(x - y) \tag{1.10}$$

for the boundary condition (1.2). The functions $L(x,y)$ and $M(x,y)$ can be calculated explicitly.

1.1. Discretization of Problem (1.8), (1.9). We divide the interval $[0,1]$ by the points $\{x_1, \dots, x_n\}$ (deflection observation points) and points $\{y_1, \dots, y_m\}$ (points of possible locations of actuators). The choice of the points of observation and possible locations of actuators can be determined by technical potentialities or can be formal (for example, if the points are equally spaced). The latter case is analogous to the choice of points in the problems of topology design.

Solutions (1.8) and (1.9) for $p(y)$ from class (1.7) become

$$u(x_i) = G(x_i) + \sum_{j=1}^{m} L_{ij} p_j, \tag{1.11}$$

where $L_{ij} = L(x_i, y_j)$ for (1.8) or $L_{ij} = M(x_i, y_j)$ for (1.9), $G(x) = \int\limits_0^1 L(x,y)F(y)\,dy$ is a known function equal to the deflection of the beam loaded by the force $F(y)$, i.e., deflection of an uncontrollable beam, and p_j is the intensity of the actuator at the point y_j ($p_j = 0$ means that there is no actuator at the point y_j).

Conditions (1.5) for functions from class (1.7) take the form

$$\sum_{j=1}^{m} p_j \leqslant 1, \qquad p_j \geqslant 0. \tag{1.12}$$

Introducing the vectors

$$\begin{gathered} \boldsymbol{u} = \{u(x_i),\ i = 1, \ldots, n\} \in \mathbb{R}^n, \\ \boldsymbol{y}_j = \{L_{ij},\ i = 1, \ldots, n\} \in \mathbb{R}^n, \quad \boldsymbol{y}_0 = \{G(x_i),\ i = 1, \ldots, n\} \in \mathbb{R}^n, \end{gathered} \tag{1.13}$$

we write (1.11) as

$$\boldsymbol{u} = \boldsymbol{y}_0 + \sum_{j=1}^{m} \boldsymbol{y}_j p_j \in \mathbb{R}^n. \tag{1.14}$$

Provided the condition

$$\sum_{j=1}^{m} p_j = 1 \qquad (p_j \geqslant 0) \tag{1.15}$$

is satisfied, the set $\sum\limits_{j=1}^{m} \boldsymbol{y}_j p_j$ forms a convex shell of points $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\}$, i.e., a polyhedron $P^0 = \operatorname{conv}\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\}$. The set $\sum\limits_{j=1}^{m} \boldsymbol{y}_j p_j$ subject to condition (1.12) is a cone P with a base P^0 and vertex 0, which can be represented as $P = \operatorname{conv}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m, 0\}$. Equality (1.14) determines a cone $K = P + \boldsymbol{y}_0$ (the cone P shifted by the vector $\boldsymbol{y}_0$) with a base $K^0 = P^0 + \boldsymbol{y}_0$.

The equality $\sum\limits_{j=1}^{m} p_j = 1$ in (1.15) means that the total intensity resource of actuators is exhausted. The inequality $\sum\limits_{j=1}^{m} p_j < 1$ means that the total resource is not exhausted (i.e., only part of intensity is required to minimize the deflection).

After discretization of condition (1.6), we obtain

$$\|\boldsymbol{u}\| = \max_i |u_i| \to \min, \tag{1.16}$$

where the vector $\boldsymbol{u}$ is determined by equality (1.14) with condition (1.12), i.e., we arrive at the problem

$$\|\boldsymbol{u}\| \to \min, \qquad \boldsymbol{u} \in K. \tag{1.17}$$

The cone K is determined above.

Thus, the problem reduces to minimization of the function $\|\boldsymbol{u}\|$ on the cone K. It is convenient to solve the problem by using the geometrical analysis rather than solving the optimization problem.

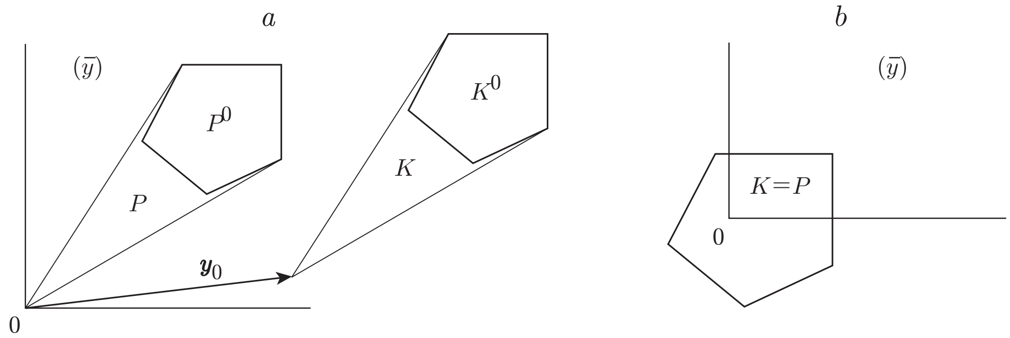

Fig. 1

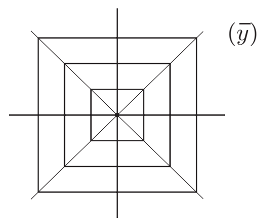

Fig. 2

We consider the following cases:

1. Point $0 \notin P^0$. In this case, the cone P does not coincide with the set P^0 (Fig. 1a) and, accordingly, the cone K does not coincide with K^0.

2. Point $0 \in P$. In this case, we have $P = P^0$ (Fig. 1b) and $K = K^0$.

By virtue of the definition of $\|\boldsymbol{u}\|$ (1.16), the condition $\|\boldsymbol{u}\| = c$ determines a cube $D(c)$ with faces $x_i = c$. For $c = 0$, the cube $D(c)$ coincides with the coordinate origin; for $c = \infty$, the cube $D(c)$ occupies the entire space $\mathbb{R}^n$ (Fig. 2).

1.2. Solution of Problem (1.17). We consider problem (1.17). It should be noted that we seek not for one optimal control but for the pair: {*optimal control of actuators*}, {*optimal location of actuators*}. These quantities are interrelated and determined simultaneously.

In finding the minimum in problem (1.17), we distinguish the following cases.

1. $0 \notin K$. If the cone K does not contain the coordinate origin, the cube $D(c)$, as it increases from the point at the coordinate origin (for $c = 0$) to $\mathbb{R}^n$ (as $c \to \infty$), touches the cone K at a certain time. The point of contact determines the solution of problem (1.17) (Fig. 3): the value of c for which the first contact occurs is the minimum in problem (1.17).

One can see from Fig. 3 that, for $0 \notin K$, two cases of contact between the cone K and the cube $D(c)$ are possible:

(a) the cube $D(c)$ touches the face of the cone K (Fig. 3a);

(b) the cube $D(c)$ touches the vertex of the cone K (one of the vertices of K is the point $\boldsymbol{y}_0$) (Fig. 3b).

In the case "a", the sum in (1.14) contains several quantities $p_{j1}, \ldots, p_{jN}$ other than zero. This implies that N actuators of intensity $p_{j1}, \ldots, p_{jN}$ located at the points $y_{j1}, \ldots, y_{jN}$ are used. If the base K^0 of the cone K touches the cube $D(c)$ (Fig. 3c), then $\sum_{j=1}^{m} p_j = 1$, i.e., the total intensity resource of actuators is used. If the set $K \setminus K^0$ touches the cube $D(c)$ (Fig. 3b), then $\sum_{j=1}^{m} p_j < 1$, i.e., the intensity resource is used incompletely.

In the case (b), in the sum in (1.14) either one quantity $p_j \neq 0$ (if the vertex differs from $\boldsymbol{y}_0$) or all the quantities $p_j = 0$ (if the vertex coincides with $\boldsymbol{y}_0$). In the first case, one actuator of intensity p_j located at the point y_j is used. In the second case, there is no control.

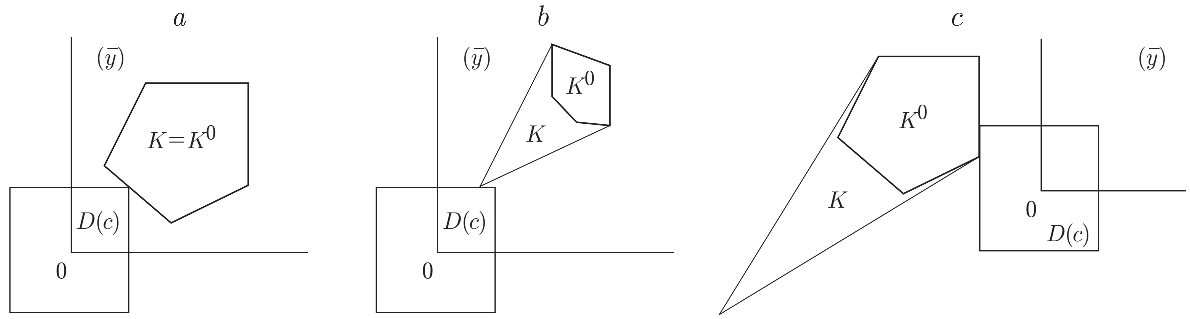

Fig. 3

2. $0 \in K$. If the cone K contains the coordinate origin, one can ensure zero deflection of the beam at the observation points. If the origin coincides with some vertex of the cone K, the sum in (1.14) contains only one nonzero term; otherwise, it contains several nonzero terms. The criterion that determines whether the intensity resource is used completely or incompletely is the same as that considered above [the cube $D(c)$ touches the set K^0 or $K \setminus K^0$].

Remark. Based on the Karateodori theorem of representation of a point belonging to a convex set in $\mathbb{R}^n$ [6], we assert that the number of actuators does not exceed $n+1$ (n is the number of deflection-control points). This estimate, however, can be substantially inflated.

1.3. Conclusions:

• The number of nonzero terms in (1.14) is equal to the number of point actuators necessary for minimization of the beam deflection, which is not determined *a priori*.

• Since $\{\boldsymbol{y}_j\}$ and $\boldsymbol{y}_0$ in (1.14) are known, the number, locations, and intensities of actuators can be determined by the method proposed above. This method allows one to determine the minimum number of actuators and their locations and intensities.

2. Continuous Problem. Case of Steady Loading. The above-considered case, where the number of points of potential locations of actuators is assumed to be finite, appears to be restrictive. We consider a more general case to show that restrictions imposed by this condition are not severe. Let the observation points be fixed and actuator locations be arbitrary. At the observation points, we obtain

$$\boldsymbol{u} = \boldsymbol{y}_0 + \int_0^1 \boldsymbol{L}(y)p(y)\,dy; \tag{2.1}$$

$$\begin{gathered} \boldsymbol{u} = \{u(x_i),\ i = 1, \ldots, n\} \in \mathbb{R}^n, \\ \boldsymbol{L}(y) = \{L(x_i, y),\ i = 1, \ldots, n\} \in \mathbb{R}^n, \quad \boldsymbol{y}_0 = \{G(x_i),\ i = 1, \ldots, n\} \in \mathbb{R}^n. \end{gathered} \tag{2.2}$$

The integral $\int_0^1 \boldsymbol{L}(y)p(y)\,dy$ with the condition $\int_0^1 p(y)\,dy = 1$, $p(y) \geqslant 0$ determines a convex shell of the line $\Gamma = \{\boldsymbol{x} = \boldsymbol{L}(y)$, where $y \in [0,1]\}$ [1], i.e., $P^0 = \operatorname{conv}\Gamma$, and the integral $\int_0^1 \boldsymbol{L}(y)p(y)\,dy$ with conditions (1.4) and (1.5) determines the cone $P = \operatorname{conv}\{\Gamma, 0\}$ with the base P^0 and vertex 0. Equality (1.14) determines the cone $K = P + \boldsymbol{y}_0$ (the cone P shifted by the vector $\boldsymbol{y}_0$) with the base $K^0 = P^0 + \boldsymbol{y}_0$.

The discretization used in Sec. 1 is the approximation of the line Γ by a polygonal line and the approximation of curvilinear cones by polyhedron cones. The approximation becomes more accurate as the number of points increases. The geometrical analysis can also be performed directly for curvilinear cones. This is a more laborious problem, although it involves no fundamental differences from the problem considered above.

3. Relation between the Location of a Force Actuator and the Point of the Maximum Deflection of the Beam. As is mentioned above, the problem is generally solved for the pair $\{$*optimal control of actuators*$\}$, $\{$*optimal location of actuators*$\}$. In particular cases, the problem is solved for one component of this pair. If the

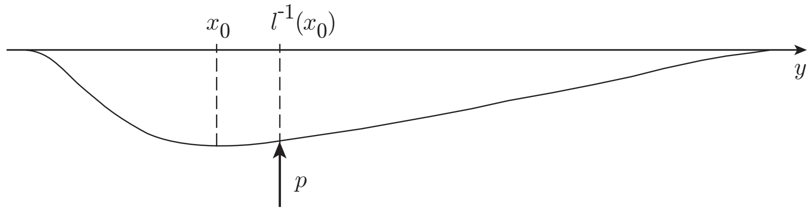

Fig. 4

location of actuators is fixed, we obtain an optimal-control problem. In the case of a beam, it is simple and is not considered here.

We consider the problem for the case where the "control" is fixed and it is required to determine the optimal location of the actuator.

Let the beam be loaded by an external force F. To decrease the deflection, it seems natural to apply a force to the point of the maximum deflection. Is this hypothesis justified? The answer follows from the results of Sec. 1: location of several actuators at different points can be optimal. This implies that, in general, the hypothesis is not valid. We consider a particular case. Let only one force actuator be used. Its location can be determined by solving the problem from Sec. 2. The solution of this problem, however, does not yield an explicit relation between the actuator location and the point of the maximum deflection. Therefore, we consider the problem from another viewpoint.

Let $L(x, y)$ be the fundamental solution corresponding to the force applied at the point y and $x^* = l(y)$ be the point of the maximum of $L(x, y)$ for given y. Accordingly, $y = l^{-1}(x^*)$, where the superscript -1 denotes the inverse function. Generally, the point of the maximum of x^* does not coincide with the point $l^{-1}(x^*)$.

Let the deflection $u(x)$ produced by the force $F(x)$ have a maximum at the point x_0 (Fig. 4). It follows from the aforesaid that the deflection is reduced more effectively if the force p is applied at the point $l^{-1}(x_0)$ rather than at the point x_0. Indeed, for the function $u(x) = G(x) + L(x, l^{-1}(x_0))p$ at the point x_0, we obtain

$$G'(x_0) = 0, \qquad G''(x_0) > 0, \tag{3.1}$$

whereas for the function $L(x_0, l^{-1}(x_0))$, we have

$$L'(x_0, l^{-1}(x_0)) = 0 \tag{3.2}$$

(the prime denotes differentiation with respect to x). For small p, by virtue of (3.1) and (3.2), we find that

$$u'(x) = G'(x_0) + L'(x_0, l^{-1}(x_0))p = 0, \qquad u''(x) = G''(x_0) + L''(x_0, l^{-1}(x_0))p > 0,$$

i.e., the maximum deflection at the point x_0 decreases. The converse is also true: for small p, the maximum remains at the point x_0 if the force is produced by the actuator located at other points.

Let the force be shifted from the point $l^{-1}(x_0)$ to a certain point y. In this case, the beam deflection is given by

$$u(x) = G(x) + L(x, y)p. \tag{3.3}$$

In the neighborhood of the point $[x = x_0,\ y = l^{-1}(x_0)]$, the first-order expansions of the derivatives of the functions $G(x)$ and $L(x, y)$ with respect to x and y have the form [with allowance for (3.1) and (3.2)]

$$G'(x) = a(x - x_0), \qquad L'(x, y) = A(x - x_0) + B(y - l^{-1}(x_0)).$$

For $u(x) = G(x) + L(x, y)p$, we have the extremum condition $u'(x) = G'(x) + L'(x, y)p = a(x - x_0) - (A(x - x_0) + B(y - l^{-1}(x_0)))p$, which implies that

$$y - l^{-1}(x_0) = a(x - x_0)(a + pA)/(pB). \tag{3.4}$$

If $x - x_0 = 0$, it follows from (3.4) that $y - l^{-1}(x_0) = 0$; if $x - x_0 \neq 0$, then $|y - l^{-1}(x_0)| \to \infty$ as $p \to 0$. Since the value of y cannot lie outside the interval $[0, 1]$, then $x = x_0$ and the maximum occurs at the point x_0. In this case, by virtue of (3.4), we obtain $y = l^{-1}(x_0)$ [i.e., the force actuator is located at the point $l^{-1}(x_0)$].

Thus, the optimal location of a single force actuator of low intensity ($p \ll 1$) is the point $l^{-1}(x_0)$. Generally, this point does not coincide with the point of the maximum deflection of the beam.

4. Variable External Loading. Determining the Location of Actuators and the Instruction for the Beam Processor. In the cases considered above, the external load $F(t)$ (which was "compensated" by actuators) was fixed. The optimal location of actuators and their necessary minimum intensity were determined uniquely. Each load had a specific set of actuators. If the load takes different values, the straightforward application of the theory from Sec. 1 may lead to the conclusion that many actuators are needed to "compensate" the load. We consider the problem of determining the minimum set of actuators.

Let there be a family of loads $F(t)$, where the parameter t belongs to the set L. Following Sec. 1, we obtain the problem

$$\|\boldsymbol{u}\| \to \min, \qquad \boldsymbol{u} \in K(t); \tag{4.1}$$

$$K(t) = \operatorname{conv}\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\} + \boldsymbol{y}_0(t) \in \mathbb{R}^n. \tag{4.2}$$

In this case, the only quantity in (4.2) that depends on t is $\boldsymbol{y}_0$ [see (1.13)].

For a given value of t, the solution of the problem can be obtained by the method described above, i.e., it depends on t as a parameter. This solution is of theoretical interest since a large number of actuators can be required. At the same time, it is obvious that, in any case, the solution has the form "*if* ⟨condition⟩, *then* ⟨action⟩." Statements (instructions) of the form "*if* ..., *then* ... " are considered as knowledge [7]. Therefore, giving these instructions to a beam means that some knowledge is imparted to the beam (which makes it an "intellectual" or "sentient" structure with some knowledge or skills). The question of gaining the knowledge from an experiment (physical or numerical) is not considered in this paper (see [8]).

The following two "intelligence" levels of the system are possible:

1. The system has many actuators or can place actuators at specified points. After problem (1.17) is solved, the system determines which actuators should be switched on and calculates their intensity t using the condition $D(c) \cap K \neq \varnothing$. In this case, the system can adapt (if the problem is solvable) to any external load F (even without specification of the family of external loads).

2. The system switches on or switches off actuators according to the rule "*if* ⟨condition for the loading parameter⟩, *then* ⟨switch on some actuators from the list⟩."

In the second case, the system should be able to identify the value of the loading parameter. For this purpose, sensors and solving of the identification problem are required. To identify the loading parameter and fulfil the instructions of Sec. 2, a processor is needed. Below, we show that these are quite simple problems; therefore, a low-power processor can control the "intellectual" structure.

First, we choose c such that $D(c) \cap K(t) \neq \varnothing$ for all t. For this value of c, the cone $K(t)$ has a common point with the cube $D(c)$ for all t. To solve the problem, it suffices to determine a set of points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M$ on the cone $K(0)$ such that at least one of the points $\boldsymbol{x}_i + \boldsymbol{y}_0(t)$ [i.e., points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M$ moving along with the cone $K(t)$] lies in $D(c)$. Moreover, it is desirable to use as few points as possible. We consider this requirement in greater detail. A point belonging to the cone K corresponds to a system of actuators. Systems of actuators corresponding to the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M$ solve the problem: if the load is such that the point $\boldsymbol{x}_i + \boldsymbol{y}_0(t) \in D(c)$, the system of actuators corresponding to the point $\boldsymbol{x}_i$ should be switched on. From the practical viewpoint, it is expedient to use few actuators, i.e., points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M$. Among these points, there should be as many vertices of the cone as possible.

The following algorithm for choosing the points is proposed:

Step 1. Determine the set of values L_0 of the loading parameter t for which $\boldsymbol{y}_0(t) \in D(c)$. In the case where $t \in L_0$ and actuators are out of operation, we have $\|\boldsymbol{u}\| < c$. There remains a set of values of the parameter $L \setminus L_0$ for which the points are not yet chosen.

Step 2. Consider the point $\boldsymbol{x}_c$ at which the cone K touches the cube $D(c)$ (this case corresponds to the worst variant) where actuators are required. Find a set of values L_c of the loading parameter for which $\boldsymbol{x}_c \in D(c)$. If there are several contact points, repeat this procedure for each contact point and include the corresponding values of the loading parameter to the set L_c. As a result, the set $L \setminus L_0 \setminus L_c$ is obtained. Further, iterations are used, which may depend on the choice of the points on the cone.

Step 3. Infer whether the point $\boldsymbol{x}_c$ belongs to:

(a) cone vertex;

(b) cone face.

Go to step 4 in the case (a) and to step 5 in the case (b).

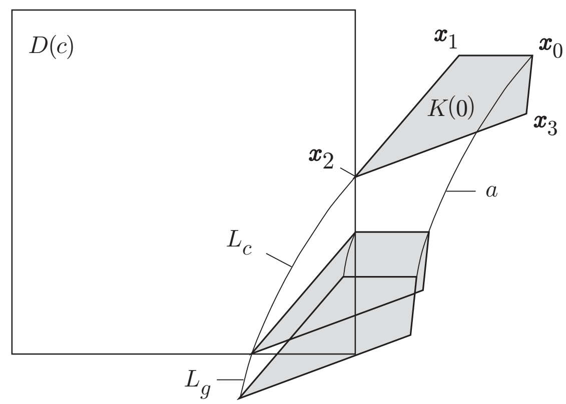

Fig. 5

Step 4. If the point $\boldsymbol{x}_c$ is the cone vertex, the following cases are possible at the moment it leaves the cube $D(c)$:

(a) a certain vertex of the cone K lies in $D(c)$. Take it as a new point $\boldsymbol{x}_c$ and go to step 3 (note, the cone vertex corresponds to one actuator, the smallest number of actuators; therefore, preference is given to the choice of the vertex);

(b) no vertices lie in the cube $D(c)$. In this case, go to step 5.

Step 5. Find the value of the loading parameter for which the face (see step 3) intersects the cube $D(c)$ and take the points that belong to the intersection of the face and cube. The points lying at one face of the cone correspond to invariable locations of actuators whose intensity varies.

Thus, the algorithm admits some arbitrariness. This arbitrariness can be useful since it allows one to solve the problem by increasing the number of actuators or by using actuators of variable intensity.

As an example, we consider a beam for which the cube $D(c)$ and the character of motion of the cone $K(t)$ are shown in Fig. 5.

In Fig. 5, the point $\boldsymbol{y}_0(t)$ lies outside $D(c)$ for all t [the curve a is the trajectory of the point $\boldsymbol{y}_0(t)$]. In this case, actuators are needed. In the interval L_c inside $D(c)$, there is a point $\boldsymbol{x}_2$ corresponding to the actuator A_2. At the moment the point $\boldsymbol{x}_2$ leaves $D(c)$ (Fig. 5), the vertex $\boldsymbol{x}_1$ enters $D(c)$ and remains in $D(c)$ until the cone K stops moving. The actuator A_1 corresponds to the vertex $\boldsymbol{x}_1$. The system is controlled according to the rule (controlling instruction for the beam): "*if* $\langle t \in L_c \rangle$, *then* $\langle$switch on the actuator A_2 of intensity $p_2 = 1\rangle$" or "*if* $\langle t \in L_g \rangle$, *then* $\langle$switch on the actuator A_1 of intensity $p_1 = 1\rangle$."

We consider a condition of the form $t \in L_c$ (the current parameter of external loading lies in a specified interval) It is convenient to determine the current value of the loading parameter t in terms of the current strains of the beam. However, the beam can be simultaneously loaded by the external load $F(t)$ and controlling actions p. We formalize this situation. Let there be a functional $\Phi(F, p)$ such that $t = \Phi(F(t), p(t))$. The existence of this functional means that the system measures the current value of the functional $\Phi(F, p)$, which is used to measure the loading parameter t (i.e., the system identifies the current loading parameter), and determines which group of actuators should be switched on. The system should have sensors for measurements and a processor for calculations and making a decision to switch on actuators. The processor performs simple operations: calculation of values of the functional Φ and implementation of the instruction of the form "*if* ..., *then* ... ." Therefore, the "intellectual" structure does not need a high-power processor.

The choice of the functional Φ (and hence, the choice of the system of sensors and the method for processing signals from them) is determined by a specific class of loads. The functional Φ can easily be constructed for a specific class of loads; it is, therefore, inexpedient to describe the general method of its construction.

Thus, to solve the problem of control of a beam by means of actuators for external loads from a certain class, it is necessary to supplement the beam–actuators system by sensors and a processor, which determine the loading parameter and switch on actuators according to the controlling instruction.

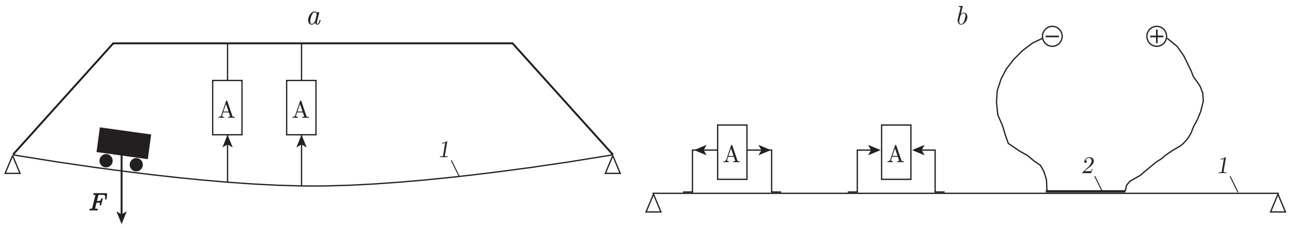

Fig. 6

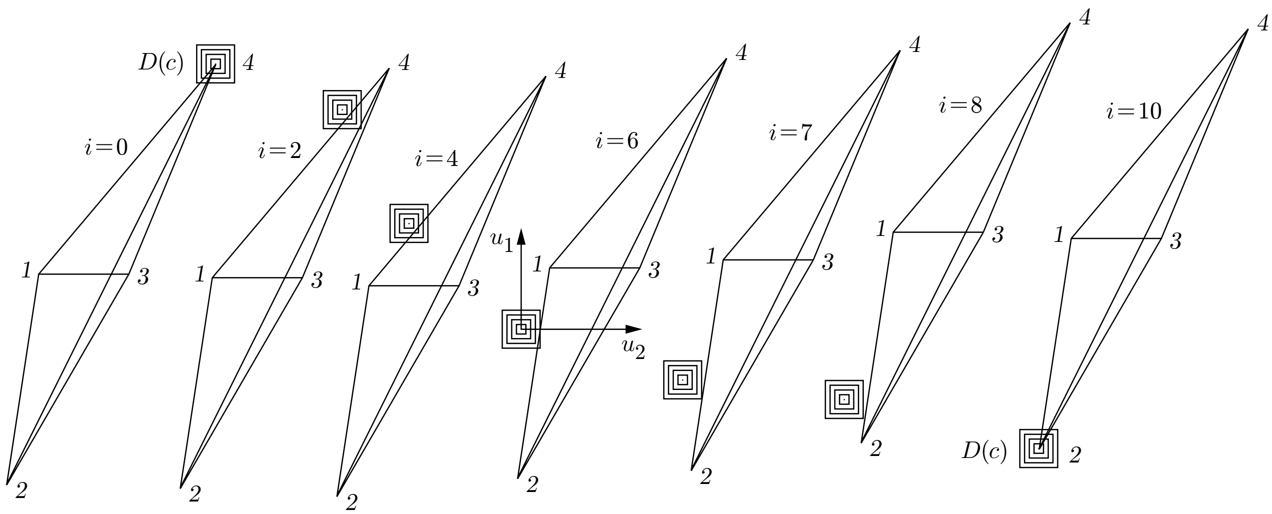

Fig. 7

It is possible to "separate" the intellectual resource: a low-power processor, which calculates the values of Φ and implements instructions of the form "*if* ..., *then* ...," is a component of the structure and performs real-time control. Another processor, which solves a more complex problem (4.1), (4.2), produces instructions of the form "*if* ..., *then* ... (knowledge)" for the first processor. This processor can be not a component of the structure and take no part in real-time control of the structure.

5. Actuators of Forces and Moments. Figure 6a and b shows possible diagrams that explain the action of actuators of forces and moments, respectively [beam (1), piezoelectric plate (2), and actuator (A)]. It should be noted that the control forces in Fig. 6a are external, whereas the moments in Fig. 6b arise as a result of the action of the beam on itself. Control of the beam by means of external forces refers largely to the optimal-control problems. In these cases, as a rule, the location of control forces depends on external surroundings of the structure. The control where the structure acts on itself refers to "intellectual" structures. Indeed, the loading schemes shown in Fig. 6b can be installed at arbitrary points of the beam, and restrictions on their installation usually depend on the structure of the beam (rather than on external surroundings).

As an example, we consider a beam $[0, 1]$ with the ends $y = 0$ and $y = 1$, loaded by an external point force moving between the points $x_1 = 0$ and $x_2 = 0.5$ (Fig. 6a). Let the deflections be observed at the points $x_1 = 0.25$ and $x_2 = 0.5$. We determine possible locations of actuators at the points $y_1 = 0.25$, $y_2 = 0.5$, and $y_3 = 0.75$. Figure 7 shows the evolution of the cone $K(t)$ for the problem considered; the values $t = 0.05i$ $(i = 0, 1, \ldots, 10)$ correspond to the location of the external load moving from the beam end (0) to its middle (0.5) with a step of 0.05. The squares in Fig. 7 refer to the cube $D(c)$. In the two-dimensional case considered, $D(c)$ is a square obtained by the method outlined in Sec. 1. The triangle 123 is K^0 and the triangle 124 is $K \setminus K^0$. The point 4 refers to the value of $\boldsymbol{y}_0(t)$.

It follows from Fig. 7 that, for $i \leqslant 5$, the minimum deflections can be maintained by one actuator A_1 located at the point $y_1 = 0.25$, whose intensity increases as the point of application of the force $F(t)$ moves from 0 to the middle of the beam. For $i = 6$, 7, and 8, it is necessary to use the actuator A_1 in combination with the actuator

TABLE 1

i	p_1	p_2	i	p_1	p_2
0	0	0	6	0.67	0.31
1	0	0	7	0.33	0.67
2	0.23	0	8	0.11	0.89
3	0.47	0	9	0	1.00
4	0.74	0	10	0	1.00
5	1.00	0			

A_2 (at the point $y_2 = 0.5$). For $i = 9$ and 10, one can use all three actuators or only the actuators A_1 and A_2. At the point $i = 10$, one can use only the actuator A_2. Beginning from $i = 6$, the actuators use the total intensity completely (which follows from $D(c) \cap K^0 = \varnothing$ and $D(c) \cap (K \setminus K^0) \neq \varnothing$ for given values of i). We note that the use of only one actuator A_2 (located at the middle of the beam) may fail to minimize the deflection. In Fig. 7, the action of the actuator A_2 with different intensities is shown by curve 2–4. For all values of i (but $i = 9$ and 10), this curve does not intersect $D(c)$, i.e., only one actuator A_2 cannot ensure the optimal control.

Calculation of the intensity of actuators reduces to calculation of proportions of the sectors shown in Fig. 7. As a result, we obtain the following instruction for controlling the beam by means of two actuators A_1 and A_2 (see Table 1).

It should be noted that the intensities (except for the case of $i = 7$) are determined nonuniquely because of the nonunique choice of the point from $D(c) \cap K(t)$. The differences in the values of intensities are proportional to the size of the square $D(c)$. For the controlled beam, the maximum deflection occurs in the case $i = 7$ (the force is applied at the point $y = 0.35$ and the maximum deflection is equal to c). In this case, the cone $K(t)$ touches the cube $D(c)$. In other cases, the cube and cone intersect at internal points. The maximum deflection of the free beam u_f occurs if the force is applied at the point $x = 0.5$ and $u_f/c \approx 21$.

Thus, the way the cone $K(t)$ passes through the square $D(c)$ determines both mechanical elements of the "intellectual" beam (number and location of actuators) and its logical elements (controlling program). The number of the points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_M$ in the minimum representation $\boldsymbol{x}_c = \sum_{j=1}^{m} p_j(t)(\boldsymbol{x}_j + \boldsymbol{y}_0(t))$ is equal to the minimum number of actuators and their subscripts indicate the location of actuators. The controlling instruction is formed from the numbers $p_j(t)$.

6. Continuous Problem. Variable External Loading. As in Sec. 2, we fix the observation points and do not fix locations (even potentially possible) of actuators. At the observation points $\{x_i,\ i = 1, \dots, n\}$, we have

$$\boldsymbol{u} = \boldsymbol{y}_0(t) + \int_0^1 \boldsymbol{L}(y) p(y)\, dy; \tag{6.1}$$

$$\boldsymbol{u} = \{u(x_i),\ i = 1, \dots, n\} \in \mathbb{R}^n,$$
$$\boldsymbol{L}(y) = \{L(x_i, y),\ i = 1, \dots, n\} \in \mathbb{R}^n, \quad \boldsymbol{y}_0(t) = \{G(x_i),\ i = 1, \dots, n\} \in \mathbb{R}. \tag{6.2}$$

Problem (6.1), (6.2) is similar to the problem considered in Sec. 2 but differs in that the quantity $\boldsymbol{y}_0$ depends on t. Equality (6.1) subject to constraints (1.4) and (1.5) specifies the moving cone $K(t) = P + \boldsymbol{y}_0(t)$ [the cone $P = \operatorname{conv}\{\Gamma, 0\}$ shifted by the vector $\boldsymbol{y}_0(t)$].

The approach based on the analysis of convex shells allows one to solve the problem of determining the number, location, and controlling instruction of actuators. The set of possible displacements is $K(t) = \operatorname{conv}\{\Gamma, 0\} + \boldsymbol{y}_0(t)$. As t varies, the cone $K(t) = \operatorname{conv}\{\Gamma, 0\} + \boldsymbol{y}_0(t)$ moves. In the process, it always contains the point 0. This point also belongs to the line Γ [which corresponds to the solution $p(y, t) = -F(y, t)$]. In this case, the shape of the beam remains unchanged (deflection is equal to zero), but the number of actuators must be infinite. We assume that the deflection $\|\boldsymbol{u}\| \leqslant c$ is possible and infer: whether a finite number of actuators can be used in this case; how to choose these actuators.

It is worth noting that, in this case, possible locations of actuators are not specified. As is shown above, the number and location of actuators are determined by the location of the cone $K(t)$ and cube $D(c)$ relative to each other. In this case, it is convenient to consider displacement of the cube relative to the cone.

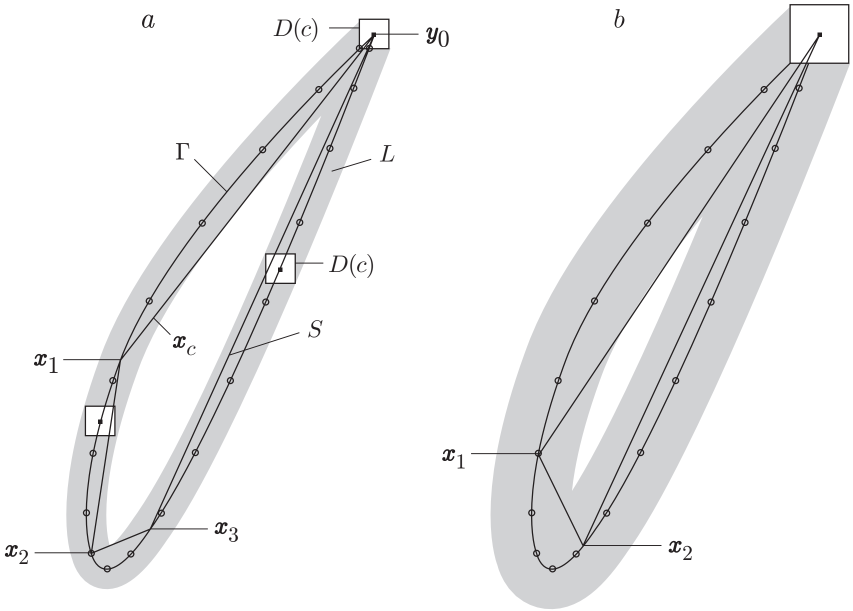

Fig. 8

Let us describe the solution method (which is universal) using the following example. Figure 8 shows the line Γ for $\boldsymbol{L}(y) = \{L(0.25, y),\, L(0.5, y),\, y \in [0, 1]\} \in \mathbb{R}^2$, which corresponds to an external point force moving from the end $y = 0$ to the end $y = 1$ (see Fig. 6a) and sensors located at the points $x_1 = 0.25$ and $x_2 = 0.5$. Since deflections are allowed, the solution is determined by the passage of the cone $K(t)$ through the cube $D(c)$ located at the coordinate origin. One can consider the motion of the cube $D(c)$ relative to the cone $K(t)$, which is more convenient in this case. In this example, the line Γ is closed and convex and contains the point 0. The line Γ was calculated using a computer. All the calculations were also performed on a computer. By virtue of convexity of the line Γ, we obtain $P = \operatorname{conv}\{\Gamma, 0\} = \operatorname{conv}\Gamma$. The square [in this case, the cube $D(c)$ is a square] moves along the curve Γ. Figure 8 shows the squares that correspond to various admissible deflections c. The admissible deflections in Fig. 8b are twice as large as those in Fig. 8a. The center of the square $D(c)$ (corresponding to the point 0) moves along the curve Γ. Upon variation of t from 0 to 1, the square travels along the entire line and "sweeps" a certain tube L. The squares in the upper part of Fig. 8 correspond to the values $t = 0$ and $t = 1$ [the squares $D(c)$ coincide for $t = 0$ and $t = 1$].

Possible displacements due to the action of actuators are represented by convex combinations of points lying on the line Γ. Let us find the minimum number of actuators that can maintain beam deflections (at the observation points) within a specified interval. To solve this problem, it is necessary to find the minimum number of points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ such that $L \cap \operatorname{conv}\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\} \neq \emptyset$, i.e., it is required to find a polygonal line S with the minimum number of segments inscribed into the line Γ (i.e., with vertices on Γ) and lying entirely inside the tube L. This polygonal line can easily be constructed. The construction (whose principle is seen from Fig. 8a) begins at the point $\boldsymbol{y}_0$. A straight line belonging to the tube L is drawn from the point $\boldsymbol{y}_0$ to the maximally distant point $\boldsymbol{x}_1$ of the curve Γ. The first actuator is placed at the point of the beam corresponding to $\boldsymbol{x}_1$. Similar construction is performed beginning from the point $\boldsymbol{x}_1$ and so on over the entire tube L. As a result, the desired polygonal line is constructed. Figure 8a corresponds to the case where the polygonal line has three vertices $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ at which actuators are placed. Figure 8b corresponds to the case where the polygonal line has two vertices $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The vertex $\boldsymbol{y}_0$ corresponds to the uncontrolled beam. Locations of actuators can be determined by the following procedure. We define the line Γ in a parametric form using the beam length as a parameter. In our case, t plays the role of the parameter. In Fig. 8, the points on the line Γ are specified with a step equal to 1/20 of the beam length. The actuators A_1, A_2, and A_3 are located at the points $x_1 \approx 0.29$, $x_2 \approx 0.45$, and $x_3 \approx 0.58$, respectively.

Thus, we have determined the minimum number of actuators that maintain deflections within specified limits and actuator locations, i.e., we have solved the part of the problem associated with structural design. The next step is to develop the logical part of the project, namely, the instruction for controlling the actuators. The instruction is written with the help of Fig. 8 in the following manner. For an arbitrary value $t = t_0$, the square $D(c)$ and polygonal line S have a common point $\boldsymbol{x}_c$. The point $\boldsymbol{x}_c$ is represented in the form of a convex combination of vertices of the polygonal line S. For the point $\boldsymbol{x}_c$ shown in Fig. 8, we have $\boldsymbol{x}_c = p_2\boldsymbol{x}_2 + p_3\boldsymbol{x}_3$. Hence, we obtain the instruction "*if* $\langle t = t_0\rangle$, *then* $\langle p_1(t) = 0$, $p_2(t) = p_2$, and $p_3(t) = p_3\rangle$." Similar calculations should be performed for all points on the beam (for a finite number of points spaced with a small step). As a result, we obtain the complete instruction for controlling the actuators.

Since the choice of the point $\boldsymbol{x}_c$ and its representation in the form of a convex combination of the points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, and $\boldsymbol{y}_0$ are nonunique, the instruction is also nonunique. However, all instructions differ (for a given set of actuators) only slightly. The differences in actuator intensities are proportional to the width of the tube L, i.e., admissible deflection of the beam c. The locations of the minimum set of actuators are also determined nonuniquely. The construction of the polygonal line shown in Fig. 8a was performed in the counterclockwise direction. For the clockwise direction, other locations of actuators are obtained.

Figure 8b shows the solution of the same problem with a twofold increase in the admissible deflection c. In this case, it suffices to locate only two actuators at the points $x_1 = 0.35$ and $x_2 = 0.56$.

7. Generalization of the Problem. First, it should be noted that the method of the solution proposed is applicable to any structure whose behavior under loading can be described by influence functions [of the form (1.8) and (1.9)]. Most of elastic structures meet this requirement [9].

An increase in the number of observation points increases the dimensionality of the problem. In this case, the principle of constructing the solution remains unchanged but other mathematical methods of the solution are required. The reason is that we cannot use direct geometrical methods of the solution (see the examples given above) and have to resort to computational algorithms implementing these methods in high-dimensional spaces. Methods for developing these algorithms are not always trivial and are studied in computational geometry (see [10]).

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